RELATION BETWEEN TURÁN EXTREMUM PROBLEM AND VAN DER CORPUT SETS

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Abstract. Let $K \subset \mathbb{N}$ and $\mathbf{T}(K)$ is a set of trigonometric polynomials

$$T(x) = T_0 + \sum_{k\in K,\, k\le H} T_k \cos(2\pi kx), \qquad H > 1,$$

$T(x) \ge 0$ for all x and $T(0) = 1$.

Suppose that $0 < h \le 1/2$ and $K(h)$ is the class of functions

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(2\pi nx)$$

satisfying the following conditions: $a_n \ge 0$ for all n, $f(0) = 1$ and $f(x) = 0$ for $h \le |x| \le 1/2$.

We consider an relation between extremum problem

$$\delta(K) = \inf_{T\in\mathbf{T}(K)} T_0$$

and Turán extremum problem

$$A(h) = \sup_{f\in K(h)} a_0 = \sup_{f\in K(h)} \int_{-h}^{h} f(x)\,dx$$

for rational numbers $h = p/q$ and set $K = \bigcup_{\nu=0}^{\infty}\{q\nu + p, \dots, q\nu + q - p\}$.

The problem $\delta(K)$ is connection with van der Korput sets. Van der Korput sets study in analytic number theory.

In number theory following question is important (see [1], we use results of Chapter 2): is given set of numbers

$$\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}$$

uniformly distributed?

To date several methods are known for solving this problem [1]. One way was given by J.G. van der Corput. This method consists in research of sequence

$$\{u_{n+k} - u_n\}_{n=1}^{\infty},$$

where k runs over some set K of natural numbers.

Definition ([1]). A set $K \subset \mathbb{N}$ is called a van der Corput set if the sequence $\{u_n\}_{n=1}^{\infty}$ is uniformly distributed (mod 1) whenever the differenced sequence

$$\{u_{n+k} - u_n\}_{n=1}^{\infty}$$

is uniformly distributed (mod 1) for all $k \in K$.

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Set of natural numbers $K = \mathbb{N}$ is a van der Corput set. It was proved by van der Corput [1].

In [1], criterion was given for determining whether a set $K \subset \mathbb{N}$ is a van der Corput set. It is founded on following extremum problem for positive trigonometric polynomials.

Problem 1. Let $\mathbf{T}(K)$ be a set of trigonometric polynomials $T(x)$ such as following conditions are satisfied:

1) $T(x) = T_0 + \sum\limits_{\substack{k\in K\\ k\le H}} T_k \cos(2\pi kx),\ H > 1;$

2) $T(x) \geq 0\ \forall x \in \mathbb{R};$

3) $T(0) = 1.$

One needs to find value

$$\delta(K) = \inf_{T\in\mathbf{T}(K)} T_0.$$

Theorem ([1])**.** *A set $K \subset \mathbb{N}$ is a van der Corput set if and only if*

$$\delta(K) = 0.$$

From this it follows that $\delta(\mathbb{N}) = 0$ (van der Cortup's result) and $\delta(K) > 0$ for any finite set K. Therefore a van der Corput set is not finite.

Exact value of $\delta(K)$ is known in few cases. Now, two examples of exact values [1].

Example 1. Suppose $q \in \mathbb{N}$, $q \geq 2$,

$$K_q^0 = \{1, 2, \dots, q-1\}, \qquad K_q = q\mathbb{Z}_+ + K_q^0 = \{k \in \mathbb{N} : q \nmid k\},$$

here $q\mathbb{Z}_+ + K = \{q\nu + k : \nu \in \mathbb{Z}_+,\ k \in K\}$. Then

$$\delta(K_q^0) = \delta(K_q) = \frac{1}{q}.$$

Thus this set K_q is not a van der Corput set.

Example 2. $\delta(\{2,3\}) = \dfrac{\cos(\pi/5)}{1+\cos(\pi/5)} = 0{,}44721\dots$

Many properties are established for value $\delta(K)$ in [1]. Now, some of them (here $K, K_1, K_2 \subset \mathbb{N}$, $q \in \mathbb{N}$):

1) if $K_1 \subset K_2$, then $\delta(K_1) \geq \delta(K_2)$;

2) $\delta(qK) = \delta(K)$, where $qK = \{qk : k \in K\}$;

3) $\delta(K^{(q)}) \leq q\delta(K)$, where $K^{(q)} = \{k \in K : q \mid k\}$ (if set $K^{(q)}$ is empty, then $\delta(K) \geq 1/q$);

4) $\delta(K_1)\delta(K_2) \leq \delta(K_1 \cup K_2)$.

These properties show that it is desirable to know value $\delta(K)$ for given set K even if $\delta(K) \neq 0$. Examples from [1]: set $Q = \{\nu^2+1\}_{\nu=1}^{\infty}$ and set of prime numbers $\mathbb{P}$ are not van der Corput sets, since $Q \subset K_3$, $\mathbb{P} \subset K_4$ and $\delta(Q) \geq 1/3$, $\delta(\mathbb{P}) \geq 1/4$.

There exists a relation between extremum problem $\delta(K)$ and Turán extremum problem $A(h)$ [2] for rational numbers $h = p/q$.

Turán's problem ([2])**.** Suppose that $0 < h \leq 1/2$ and $K(h)$ is the class of continuous 1-periodic even functions $f(x)$ satisfying the following conditions:

1) $f(x) = \sum\limits_{n=0}^{\infty} a_n \cos(2\pi n x)$;

2) $a_n \geq 0$ for all $n = 0, 1, 2, \ldots$;

3) $f(0) = \sum\limits_{n=0}^{\infty} a_n = 1$;

4) $f(x) = 0$ for $h \leq |x| \leq 1/2$.

It is required to evaluate the quantity

$$A(h) = \sup_{f \in K(h)} a_0 = \sup_{f \in K(h)} \int_{-h}^{h} f(x)\,dx.$$

Let $p,\, q \in \mathbb{N},\ 2p \leq q,\ (p, q) = 1$. In 1972 S.B. Stechkin (see [2]) solved the problem for $p = 1,\ q = 2, 3, \ldots$ $(A(1/q) = 1/q)$. In [2], the value of $A(h)$ was calculated for $p = 2,\ 3$ and $q = 2p + 1$.

Consider sets

$$K^0_{p,q} = \{p, p+1, \ldots, q-p\}, \qquad K_{p,q} = q\mathbb{Z}_+ + K^0_{p,q}.$$

Theorem 1. *For $p = 1,\ q = 2, 3, \ldots$*

$$\delta(K^0_{1,q}) = \delta(K_{1,q}) = A(1/q) = \frac{1}{q}.$$

For $p = 2,\ q = 3, 5, \ldots$

$$\delta(K^0_{2,q}) = \delta(K_{2,q}) = A(2/q) = \frac{1 + \cos(\pi/q)}{q \cos(\pi/q)}.$$

For $p = 3,\ q = 7, 8, 10, 11, \ldots$

$$\delta(K^0_{3,q}) = \delta(K_{3,q}) = A(3/q) = \frac{1}{q}\Big(1 + \frac{1 - 2(\cos(2\pi r_0/q) + \cos(2\pi(r_0+1)/q))}{1 + 2\cos(2\pi r_0/q)\cos(2\pi(r_0+1)/q)}\Big),$$

where $r_0 = [q/3]$ is an integer part of $q/3$.

For $q = 2p + 1,\ p = 1, 2, \ldots$

$$\delta(K^0_{p,2p+1}) = \delta(K_{p,2p+1}) = A(p/(2p+1)) = \frac{\cos(\pi/(2p+1))}{1 + \cos(\pi/(2p+1))}.$$

Since $K^0_{1,q} = K^0_q,\ K_{1,q} = K_q,\ K^0_{2,3} = \{2, 3\}$ and

$$A(1/q) = \frac{1}{q}, \qquad A(2/5) = \frac{1 + \cos(\pi/5)}{5\cos(\pi/5)} = \frac{\cos(\pi/5)}{1 + \cos(\pi/5)},$$

then by theorem 1, so that we get examples 1 and 2.

Proof of theorem 1 in the case $p = 3$. Let $f(x)$ be arbitrary function of class $K(p/q)$, and let $T^{(\varepsilon)} \in \mathbf{T}(K_{p,q})$ be polynomial such that

$$T^{(\varepsilon)}_0 \leq \delta(K_{p,q}) + \varepsilon, \tag{1}$$

where $\varepsilon > 0$ is a little number. Since $T^{(\varepsilon)}_k = 0$ for $k \in \mathbb{N} \setminus K_{p,q},\ f(k/q) = 0$ for $k = q\nu + k'$ $(\nu \in \mathbb{Z}_+,\ k' = p, \ldots, q - p)$, i.e. for $k \in K_{p,q}$ (by definition of set $K_{p,q}$), it follows that

$$T^{(\varepsilon)}_0 = \sum_{\substack{k \in K_{p,q} \cup \{0\} \\ k \leq \deg T^{(\varepsilon)}}} T^{(\varepsilon)}_k f(k/q) = \sum_{n=0}^{\infty} a_n \sum_{\substack{k \in K_{p,q} \cup \{0\} \\ k \leq \deg T^{(\varepsilon)}}} T^{(\varepsilon)}_k c(nk/q) = \sum_{n=0}^{\infty} a_n T^{(\varepsilon)}(nk/q).$$

Hence by nonnegativeness of a_n $\forall n \in \mathbb{N}$ and $T^{(\varepsilon)}(x)$ $\forall x \in \mathbb{R}$, by equality $T^{(\varepsilon)}(0) = 1$ and inequality (1), so that

$$a_0 \le \delta(K_{p,q}) + \varepsilon.$$

Since this inequality is established for arbitrary function $f \in K(p/q)$, it follows that

$$\sup_{f\in K(p/q)} a_0 = A(p/q) \le \delta(K_{p,q}) + \varepsilon.$$

Hence we obtain lower estimate for value $\delta(K_{p,q})$ as $\varepsilon \to 0$:

$$A(p/q) \le \delta(K_{p,q}).$$

Let us show that $\delta(K^0_{p,q}) \le A(p/q)$ for $p = 3$, $q = 7, 8, 10, 11, \ldots$ Suppose

$$\Gamma(\nu) = \gamma_0 + \gamma_1 \cos(2\pi r_0\nu/q) + \gamma_2 \cos(2\pi(r_0 + 1)\nu/q), \qquad r_0 = [q/3], \quad \nu \in \mathbb{Z},$$

where the coefficients γ_i choose from the equations

$$\Gamma(0) = 1,\ \Gamma(1) = \Gamma(2) = 0 \iff \begin{cases} \gamma_0 + \gamma_1 + \gamma_2 = 1, \\ \gamma_0 + \gamma_1 \cos(2\pi r_0/q) + \gamma_2 \cos(2\pi(r_0 + 1)/q) = 0, \\ \gamma_0 + \gamma_1 \cos(4\pi r_0/q) + \gamma_2 \cos(4\pi(r_0 + 1)/q) = 0. \end{cases}$$

In [2] prove that $\gamma_i > 0$, $i = 0, 1, 2$, and

$$\frac{1}{q\gamma_0} = A(3/q). \tag{2}$$

Let $F(x) = F_q(x)$ is Fejér's polynomial

$$F(x) = \sum_{\nu=0}^{q-1} F_\nu \cos(2\pi\nu x) = \frac{1}{q}\Big(1 + 2\sum_{\nu=1}^{q-1}\Big(1 - \frac{\nu}{q}\Big)\cos(2\pi\nu x)\Big) = \Big(\frac{\sin(\pi q x)}{q\sin(\pi x)}\Big)^2.$$

Consider polynomial $T^*(x)$

$$\begin{aligned} T^*(x) = F(x) &+ \frac{\gamma_1}{2\gamma_0}\left(F(x + r_0/q) + F(x - r_0/q)\right) + \\ &+ \frac{\gamma_2}{2\gamma_0}\left(F(x + (r_0 + 1)/q) + F(x - (r_0 + 1)/q)\right) = \\ = \sum_{\nu=0}^{q-1} &\gamma_0^{-1} F_\nu(\gamma_0 + \gamma_1\cos(2\pi r_0\nu/q) + \gamma_2\cos(2\pi(r_0 + 1)\nu/q))\cos(2\pi\nu x) = \\ &= \sum_{\nu=0}^{q-1} \gamma_0^{-1}\Gamma(\nu)F_\nu\cos(2\pi\nu x). \end{aligned} \tag{3}$$

The polynomial $T^*(x)$ satisfies conditions 1)–3) of set $\mathbf{T}(K^0_{3,q})$.
Since $\Gamma(\nu) = \Gamma(q - \nu) = 0$ as $\nu = 1, 2$, $\Gamma(0) = 1$, $F_0 = 1/q$, it follows that

$$T^*(x) = \frac{1}{q\gamma_0} + \sum_{\nu=3}^{q-3} \gamma_0^{-1}\Gamma(\nu)F_\nu\cos(2\pi\nu x) = T^*_0 + \sum_{k\in K^0_{3,q}} T^*_k\cos(2\pi\nu x).$$

Therefore the polynomial $T^*(x)$ satisfies condition 1) of set $\mathbf{T}(K^0_{3,q})$.

All coefficients $\gamma_i > 0$ and $A(3/q) = (q\gamma_0)^{-1}$ (2). Hence, by positiveness of polynomial $F(x)$ and definition of polynomial $T^*(x)$ (3), so that $T^*(x) \ge 0$ $\forall x \in \mathbb{R}$. The condition 2) is satisfied.

Now we verify the condition 3). We have

$$T^*(0) = F(0) + \frac{\gamma_1}{\gamma_0} F(r_0/q) + \frac{\gamma_2}{\gamma_0} F((r_0+1)/q).$$

However $F(\nu/q) = 0$ for $\nu = 1, 2, \ldots, q-1$, and points r_0, $r_0 + 1$ are integers from an interval $(0, q/2)$. Therefore $F(r_0/q) = F((r_0+1)/q) = 0$ and

$$T^*(0) = F(0) = 1.$$

Thus $T^*(x)$ belongs to set $\mathbf{T}(K^0_{3,q})$ and

$$T^*_0 = \frac{1}{q\gamma_0} = A(3/q).$$

Hence upper estimate is $\delta(K^0_{3,q}) \le T^*_0 = (q\gamma_0)^{-1} = A(3/q)$.

Finally using $K^0_{p,q} \subset K_{p,q}$ and property 1) of $\delta(K)$, we get estimates

$$A(p/q) \le \delta(K_{p,q}) \le \delta(K^0_{p,q}) \le A(p/q),$$

i.e.

$$\delta(K^0_{p,q}) = \delta(K_{p,q}) = A(p/q).$$

Polynomial $T^*(x)$ belongs to set $\mathbf{T}(K^0_{p,q})$ and belongs to set $\mathbf{T}(K_{p,q}) \supset \mathbf{T}(K^0_{p,q})$. It is extremal polynomial. This completes the proof of theorem. □

References

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